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ORIENTED LOCAL ENTROPIES FOR EXPANSIVE ACTIONS BY COMMUTING AUTOMORPHISMS*

BY

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ABSTRACT

Given an expansive action α of $\mathbb{Z}^2$ by automorphisms of a compact connected metrizable abelian group X, we show how the entropy of the action may be decomposed into local contributions,

$$h(\alpha) = \sum_{p \leq \infty} h_p^{(a,b)}(\alpha) \tag{1}$$

in which the summand $h_p^{(a,b)}(\alpha)$ represents the p-adic entropy due to arithmetic or geometric hyperbolicity in the direction (a,b). We recognize the p-adic contribution as an integral over the p-adic unit circle, in analogy with the global counterpart. As (a,b) changes, the decomposition (1) changes only when the line through (a,b) passes through one of a finite collection of critical directions, which are explicitly identified.

1. Introduction

Let α be an ergodic automorphism of a compact metrizable group X. By Berg's theorem [Be], Haar measure on X is maximal for α, so we may speak of the entropy of α, $h(\alpha)$, to mean either the topological or the metric entropy. Yuzvinskii, [Y], has shown how $h(\alpha)$ may be decomposed into three contributions

$$h(\alpha) = h(\alpha_{X/X^\circ}) + h(\alpha_{X^\circ/Z}) + h(\beta) \tag{1.1}$$

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where X^o is the connected component of the identity in X, and Z is the centre of X^o. The possible values of the first two terms are the logarithm of an integer or infinity, and zero or infinity respectively. The third term corresponds to an automorphism β of a compact connected abelian group Z. For the case where Z is finite dimensional (in which case it is a **solenoid**), Yuzvinskii has computed the entropy $h(\beta)$. To explain his formula, notice first that $\widehat{Z}$ is a subgroup of $\mathbb{Q}^d$ for some d; it follows that $\widehat{\beta}$ is determined by an invertible rational matrix $B \in \mathrm{GL}(d, \mathbb{Q})$. Then

$$h(\beta) = \log s + \sum_{i=1,\dots,d} \log^+ |\lambda_i| \tag{1.2}$$

where $\{\lambda_i\}$ is the set of eigenvalues of B and s is the least common multiple of the denominators of the coefficients of the characteristic polynomial of B.

In [LW], Yuzvinskii's formula is decomposed into local contributions arising from arithmetic hyperbolicity. In order to describe this, assume that $\widehat{Z} = \mathbb{Q}^d$. Then the automorphism β of $\widehat{\mathbb{Q}}^d$ may be lifted to an automorphism $\tilde{\beta}$ of $\mathbb{Q}^d_{\mathbb{A}}$,

$$\begin{array}{ccc} \mathbb{Q}^d_{\mathbb{A}} & \xrightarrow{\tilde{\beta}} & \mathbb{Q}^d_{\mathbb{A}} \\ \downarrow & & \downarrow \\ \widehat{\mathbb{Q}}^d & \xrightarrow{\beta} & \widehat{\mathbb{Q}}^d \end{array} \tag{1.3}$$

in a manner analogous to the covering $\mathbb{R}^d \to \mathbb{T}^d$ of homeomorphisms of the d-torus. This covering space argument gives a further decomposition of Yuzvinskii's formula (1.2),

$$h(\beta) = \sum_{p \le \infty} \sum_{i=1,\dots,d} \log^+ |\lambda_{i,p}|_p \tag{1.4}$$

where $\{\lambda_{i,p}\}$ is the set of eigenvalues of B in a splitting field for the characteristic polynomial of B above $\mathbb{Q}_p$, and the sum is taken over all the inequivalent completions of $\mathbb{Q}$.

Now consider a pair of commuting automorphisms of a compact group X, viewed as an action $\alpha\colon \mathbb{Z}^2 \to \mathrm{Aut}(X)$ of $\mathbb{Z}^2$. Haar measure is again maximal (see [LSW], Section 6), though it is no longer uniquely so in general. The analogous decomposition to (1.1) is shown in [LSW]

$$h(\alpha) = h(\alpha_{X/X^o}) + h(\alpha_{X^o/Z}) + h(\beta) \tag{1.5}$$

where X^o is the connected component of the identity in X, and Z is the centre of X^o. The third term is again an action of $\mathbb{Z}^2$ on a compact connected abelian group. In [LSW] the entropy of such an action is ultimately expressed in terms of certain algebraically defined constituent actions which we now describe. Let $\mathfrak{R} = \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$, and let $\mathfrak{p}$ be a prime ideal in $\mathfrak{R}$. Consider the commuting automorphisms $\times u_1$, $\times u_2$ on the $\mathfrak{R}$–module $\mathfrak{M} = \mathfrak{R}/\mathfrak{p}$. This module determines a $\mathbb{Z}^2$ action $\alpha^{\mathfrak{M}}$ by defining $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ to be the automorphisms of $X_{\mathfrak{M}} = \widehat{\mathfrak{M}}$ dual to $\times u_1$, $\times u_2$ respectively. For such an action, the entropy is computed in [LSW]: $h(\alpha^{\mathfrak{R}/\mathfrak{p}}) = 0$ if $\mathfrak{p}$ is not principal, $h(\alpha^{\mathfrak{R}/\mathfrak{p}}) = \infty$ if $\mathfrak{p} = \{0\}$, and

$$h(\alpha^{\mathfrak{R}/\mathfrak{p}}) = \int_0^1 \int_0^1 \log |f(e^{2\pi i s_1}, e^{2\pi i s_2})| ds_1 ds_2 \tag{1.6}$$

if $\mathfrak{p} = \langle f \rangle$ for some $f \neq 0$. In general, $h(\beta)$ in (1.5) is expressed as a limit of sums of expressions of the form (1.6) — see Section 4 below.

Our purpose here is to describe an extension of (1.4) to the case of $\mathbb{Z}^2$ actions, taking (1.6) as the starting point. There are two immediate difficulties. Firstly, the locally compact covering space argument (1.3) cannot be applied here: if the entropy is positive and X is connected, then the compact group is infinite-dimensional. Secondly, the formula (1.4) pertains to the map β as opposed to the action generated by β. We avoid the first difficulty here by restricting attention to expansive actions, which allows the dynamical system to be approximated by periodic points. The second difficulty is resolved as follows. Consider the automorphism β of $\widehat{\mathbb{Z}[\frac{1}{6}]}$ dual to multiplication by $\frac{3}{2}$. As pointed out in [LW],

$$h(\beta) = \log^+ |\frac{3}{2}|_\infty + \log^+ |\frac{3}{2}|_2 + \log^+ |\frac{3}{2}|_3 = \log \frac{3}{2} + \log 2 + 0 \tag{1.7}$$

and

$$h(\beta^{-1}) = \log^+ |\frac{2}{3}|_\infty + \log^+ |\frac{2}{3}|_2 + \log^+ |\frac{2}{3}|_3 = 0 + 0 + \log 3. \tag{1.8}$$

Here we think of (1.7) and (1.8) as a decomposition of $h(\beta)$ into oriented local entropies, for the directions $+1$ and -1 respectively, in the $\mathbb{Z}$ action generated by β. This leads to the following diagram representing the type of hyperbolicity in the dynamical system defined by β,

$$\xleftarrow{\quad\text{3-adic}\quad} \qquad \xrightarrow{\quad 2,\infty\text{-adic}\quad} \tag{1.9}$$

For an expansive $\mathbb{Z}^2$ action α of the form described above, we shall give a decomposition of $h(\alpha)$ into oriented local entropies

$$h(\alpha) = \sum_{p \le \infty} h_p^{(a,b)}(\alpha) \tag{1.10}$$

for all but finitely many directions $\frac{b}{a}$ in $\mathbb{Z}^2$. The decomposition (1.10) into local entropies will be seen to be invariant under small changes in the directions, and it is therefore constant in a collection of cones. These cones turn out to be finite in number, and they are explicitly identified.

We restrict attention to expansive actions on connected groups (for reasons explained in Remarks 1.1 and 1.2 of Section 5).

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2. Oriented entropies: cyclic case

Recall that $\mathfrak{R} = \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$ and any $\mathfrak{R}$–module $\mathfrak{M}$ defines a $\mathbb{Z}^2$ action $\alpha^{\mathfrak{M}}$ generated by the automorphisms dual to multiplication by u_1 and u_2 on the compact abelian group $X_{\mathfrak{M}} = \widehat{\mathfrak{M}}$. If $\mathfrak{M} = \mathfrak{R}/\mathfrak{p}$, $\mathfrak{p}$ a prime ideal not containing any constants, then $\alpha^{\mathfrak{M}}$ acts expansively if $V_{\mathbb{C}}(\mathfrak{p}) = \{(z_1, z_2) \in \mathbb{C}^2 | \ f(z_1, z_2) = 0$ for all $f \in \mathfrak{p}\}$ contains no point with $|z_1| = |z_2| = 1$ (see [S] for this result and the associated definitions). More generally, [S] proves that $\alpha^{\mathfrak{M}}$ is expansive if and only if $\mathfrak{M}$ is Noetherian and $\alpha^{\mathfrak{R}/\mathfrak{p}}$ is expansive for each prime ideal $\mathfrak{p}$ associated to the module $\mathfrak{M}$.

A **period** for the action $\alpha^{\mathfrak{M}}$ is a finite–index subgroup $\Lambda \subset \mathbb{Z}^2$, and the set of points with period Λ is defined by

$$\mathrm{Fix}_{\Lambda}(\alpha^{\mathfrak{M}}) = \{x \in X_{\mathfrak{M}} | \ \alpha_{\mathbf{n}}^{\mathfrak{M}}(x) = x \text{ for all } \mathbf{n} \in \Lambda\}. \tag{2.1}$$

It is clear that an expansive action has finitely many points of any given period. In [LSW], Section 7, it is shown that this quantity has a definite growth rate as $\|\Lambda\| = \mathrm{dist}(\Lambda \backslash \{0\}, \{0\}) \to \infty$,

$$h(\alpha^{\mathfrak{M}}) = \lim_{\|\Lambda\| \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log |\,\mathrm{Fix}_{\Lambda}(\alpha^{\mathfrak{M}})| \tag{2.2}$$

where $h(\alpha^{\mathfrak{M}})$ is the topological entropy of $\alpha^{\mathfrak{M}}$.

A **direction** in $\mathbb{Z}^2$ is a vector $(a, b) \in \mathbb{Z}^2$ with a and b coprime (include the directions $(\pm 1, 0)$ and $(0, \pm 1)$).

Definition 2.1: Let $\alpha^{\mathfrak{M}}$ be an expansive $\mathbb{Z}^2$ action on the compact group $X_{\mathfrak{M}}$. For each direction (a,b) and integer $N \geq 1$, let $X^{N,(a,b)}$ denote the closed subgroup

$$X^{N,(a,b)} = \{x \in X_{\mathfrak{M}} | \, \alpha^{\mathfrak{M}}_{(Nb,-Na)}(x) = x\}. \tag{2.3}$$

Define the automorphism $\beta_{N,(a,b)}$ to be the map $\alpha^{\mathfrak{M}}_{(a,b)}$ restricted to $X^{N,(a,b)}$.

LEMMA 2.2: *If $\alpha^{\mathfrak{M}}$ is expansive, then $\beta_{N,(a,b)}$ is also.*

Proof: First notice that the $\mathbb{Z}^2$ action on $X_{\mathfrak{M}}$ generated by the maps $\alpha^{\mathfrak{M}}_{(a,b)}$ and $\alpha^{\mathfrak{M}}_{(Nb,-Na)}$ is expansive (this follows easily from the criterion for expansiveness described above). It follows that there is an open neighbourhood U of the identity in $X_{\mathfrak{M}}$, for which

$$\bigcap_{n,m\in\mathbb{Z}} (\alpha^{\mathfrak{M}}_{(a,b)})^n(\alpha^{\mathfrak{M}}_{(Nb,-Na)})^m(U) = \{1\}. \tag{2.4}$$

Now $V = U \cap X^{N,(a,b)}$ is an open neighbourhood of the identity in $X^{N,(a,b)}$ in the subspace topology. Moreover, it is clear that

$$\bigcap_{n,m\in\mathbb{Z}} (\alpha^{\mathfrak{M}}_{(a,b)})^n(\alpha^{\mathfrak{M}}_{(Nb,-Na)})^m(U \cap X^{N,(a,b)}) = \bigcap_{n\in\mathbb{Z}} (\beta_{N,(a,b)})^n(V) = \{1\}, \tag{2.5}$$

showing that $\beta_{N,(a,b)}$ acts expansively on $X^{N,(a,b)}$. ∎

The automorphism $\beta_{N,(a,b)}$ is therefore an expansive automorphism of a compact group. If $X^{N,(a,b)}$ is assumed to be connected, then it follows by [L] that it is a solenoid. We assume now that $X^{N,(a,b)}$ is connected in the following sense: the quantities defined below are only defined when $X^{N,(a,b)}$ is connected. If $X_{\mathfrak{M}}$ is connected, then this is so for all but finitely many directions (a,b). By [LW], we therefore have

$$h(\beta_{N,(a,b)}) = \sum_{p\leq\infty} h_p(\beta_{N,(a,b)}) \tag{2.6}$$

where $h_p(\beta_{N,(a,b)})$ is the contribution to the entropy of $h(\beta_{N,(a,b)})$ due to arithmetic hyperbolicity above the p-adic completion of the rationals (in the cases $p < \infty$), and geometric hyperbolicity (in the case $p = \infty$).

Assume from now on that the $\mathfrak{R}$-module $\mathfrak{M}$ is of the form $\mathfrak{R}/\langle f\rangle$ for some non-constant irreducible polynomial f.

Definition 2.3: The (a,b)**-oriented local entropies** of $\alpha^{\mathfrak{M}}$ are the quantities

$$h_p^{(a,b)}(\alpha^{\mathfrak{M}}) = \lim_{N\to\infty} \frac{1}{N(a^2+b^2)} h_p(\beta_{N,(a,b)}), \quad p \in \mathbb{P}, \tag{2.7}$$

wherever the limit exists (here $\mathbb{P}$ is the set of inequivalent completions of $\mathbb{Q}$).

It is clear that the local entropies do not always exist (for instance, there is no reason *a priori* to assume that $X_{N,(a,b)}$ is connected). The next definition introduces directions in which the oriented local entropies exist.

Definition 2.4: Given $f \in \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$, the **associated cones** of f are the open cones in $\mathbb{Z}^2$ with vertex at the origin whose edges are lines orthogonal to lines joining any pair of points in the support of f.

For example, if $f(u_1,u_2) = 3 + u_1 + u_2$, then the support of f is the set $\{(0,0),(1,0),(0,1)\}$. The pairs $((0,0),(1,0))$, $((0,0),(0,1))$ and $((1,0),(0,1))$ define orthogonal lines $u_2 = 0$, $u_1 = 0$ and $u_1 = u_2$ respectively. Finally, the six associated cones of f are the open sets

$$\{(u_1,u_2)|\, u_1 > 0, u_1 > u_2\}, \quad \{(u_1,u_2)|\, u_1 > 0, u_1 < u_2, u_2 > 0\},$$
$$\{(u_1,u_2)|\, u_1 > 0, u_2 < 0\}, \quad \{(u_1,u_2)|\, u_1 < 0, u_1 > u_2\},$$
$$\{(u_1,u_2)|\, u_1 < 0, u_1 < u_2, u_2 < 0\}, \text{ and } \{(u_1,u_2)|\, u_1 < 0, u_2 > 0\}.$$

THEOREM 2.5: *The oriented local entropies form a decomposition of the entropy,*

$$h(\alpha^{\mathfrak{M}}) = \sum_{p\leq\infty} h_p^{(a,b)}(\alpha^{\mathfrak{M}}) \tag{2.8}$$

for any direction (a,b) in an associated cone of f.

In order to see this, we first establish that the sum in (2.8) is in fact finite. This allows us to permute taking limits in periods with the summation and to deduce (2.8) from (2.2). The proof that the local entropies exist in associated cones is postponed until Theorem 3.1.

LEMMA 2.6: *The sum (2.8) has only finitely many contributions.*

Proof of Theorem 2.5: Assume the local entropies exist, so there is convergence in (2.7). By (2.2) and Lemma 2.6 we have

$$\begin{aligned}\sum_{p\leq\infty} h_p^{(a,b)}(\alpha^{\mathfrak{M}}) &= \sum_{p\leq\infty} \lim_{N\to\infty} \frac{1}{N(a^2+b^2)} h_p(\beta_{N,(a,b)}) \\ &= \lim_{N\to\infty} \frac{1}{N(a^2+b^2)} \sum_{p\leq\infty} h_p(\beta_{N,(a,b)}) \\ &= \lim_{N\to\infty} \frac{1}{N(a^2+b^2)} h(\beta_{N,(a,b)}) \\ &= \lim_{N\to\infty} \frac{1}{N(a^2+b^2)} \lim_{M\to\infty} \frac{1}{M} \log|\mathrm{Fix}_M(\beta_{N,(a,b)})| \\ &= \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{MN(a^2+b^2)} \log|\mathrm{Fix}_{M(a,b)\mathbb{Z}+N(b,-a)\mathbb{Z}}(\alpha^{\mathfrak{M}})| \\ &= h(\alpha^{\mathfrak{M}}),\end{aligned}$$

which completes the proof. ∎

Notice that in the above proof, we have used (2.2) twice: once for expansive $\mathbb{Z}^2$ actions, and once for expansive $\mathbb{Z}$ actions. The normalization $\frac{1}{a^2+b^2}$ in Definition 2.3 is explained by the proof above: it is the ratio between MN and the index of the subgroup $M(a,b)\mathbb{Z} + N(b,-a)\mathbb{Z} \subset \mathbb{Z}^2$

We now turn to the proof of Lemma 2.6. An automorphism θ of a solenoid has **hyperbolicity set** $\mathbb{P}(\theta)$ when $h_p(\theta) = 0$ if and only if $p \notin \mathbb{P}(\theta)$. We aim to recognize the contributions to the global entropy as arising from the eigenvalues of large rational matrices. We shall need the following elementary observations.

[1] $\mathbb{P}(\theta_1 \times \theta_2) = \mathbb{P}(\theta_1) \cup \mathbb{P}(\theta_2)$.

[2] $\mathbb{P}(\theta^k) = \mathbb{P}(\theta)$ for all non-zero $k \in \mathbb{Z}$.

Proof of Lemma 2.6: Assume that (a,b) lies in an associated cone of f. Define a new polynomial $F(u_1,u_2) = f(u_1^{a^2+b^2}, u_2^{a^2+b^2})$. We claim that F is then a polynomial in the variables s and t, where $t = u_1^a u_2^b$ and $s = u_1^{-b} u_2^a$. To see this, it is enough to notice that the monomials $u_1^{a^2+b^2}$ and $u_2^{a^2+b^2}$ can be expressed in terms of s and t. This is clear:

$$(2.9) \qquad u_1^{a^2+b^2} = s^{-b}t^a, \qquad u_2^{a^2+b^2} = s^a t^b.$$

This amounts to a "diagonalisation" of a generalized power of $X^{N,(a,b)}$ in the following sense.

Let $\mathfrak{M} = \mathfrak{R}/\langle f\rangle$, $\mathfrak{N} = \mathfrak{R}/\langle f(u_1^{a^2+b^2}, u_2^{a^2+b^2})\rangle$, and $\mathfrak{L} = \mathbb{Z}[s^{\pm1}, t^{\pm1}]/\langle F(s,t)\rangle \cong \mathbb{Z}[u_1^{\pm1}, u_2^{\pm1}]/\langle F(u_1,u_2)\rangle$. Notice that $\mathfrak{L}$ may be viewed as an $\mathfrak{R}$-module and therefore determines a $\mathbb{Z}^2$ dynamical system. Now it is clear that

$$(\alpha^{\mathfrak{N}})^{(a^2+b^2)} \cong \alpha^{\mathfrak{M}} \times \cdots \times \alpha^{\mathfrak{M}} \tag{2.10}$$

where the left hand side denotes the $\mathbb{Z}^2$ action generated by the automorphisms $\alpha^{\mathfrak{N}}_{(a^2+b^2,0)}$ and $\alpha^{\mathfrak{N}}_{(0,a^2+b^2)}$, the right hand side is an $(a^2+b^2)^2$-fold Cartesian product, and $\cong$ denotes algebraic isomorphism (see [W] for a more general discussion of this kind of rescaling, and a detailed description of the isomorphism (2.10)). It follows that

$$\begin{aligned}\beta^{\mathfrak{M}}_{N,(a,b)} \times \cdots \times \beta^{\mathfrak{M}}_{N,(a,b)} &\cong \left(\beta^{\mathfrak{N}}_{N,(a,b)}\right)^{(a^2+b^2)}\\ &\cong \left(\beta^{\mathfrak{L}}_{N,(1,0)}\right)^{(a^2+b^2)}\end{aligned}$$

By the remarks [1] and [2] above, we deduce that

$$\mathbb{P}(\beta^{\mathfrak{M}}_{N,(a,b)}) = \mathbb{P}(\beta^{\mathfrak{L}}_{N,(1,0)}). \tag{2.11}$$

For the purposes of this Lemma, we may therefore assume without loss of generality that $(a,b) = (1,0)$. The change of variable from u_1, u_2 to s, t takes a direction in the associated cone of f to a direction in the associated cone of F, so we may assume additionally that $(1,0)$ is in an associated cone.

So consider the map $\beta^{\mathfrak{M}}_{N,(1,0)}$, where $\mathfrak{M} = \mathfrak{R}/\langle f(u_1,u_2)\rangle$, and $f(u_1,u_2) = f_0u_2^{n_0} + f_1u_1u_2^{n_1} + \cdots + f_du_1^d$, where $f_0,\dots,f_d \in \mathbb{Z}$. (It is more natural to write a general polynomial for which $(1,0)$ lies in an associated cone in the form $f_0 + f_1u_1u_2^{n_1} + \cdots + f_du_1^du_2^{n_d}$; it will be convenient for us to multiply by the monomial $u_2^{n_d}$ to make the leading u_1 coefficient constant.)

Given a polynomial $g(u_2) \in \mathbb{Z}[u_2]$, with $g(u_2) = g_0 + g_1u_2 + \cdots + g_ku_2^k$, define the N-**circulant** of g to be the $N \times N$ integer circulant matrix whose first row is $(g_0,\dots,g_k,0,\dots,0)$.

Now

$$\frac{\mathfrak{R}}{\langle 1-u_2^N\rangle} \cong \mathbb{Z}[u_1^{\pm1}]^N \tag{2.12}$$

(as $\mathbb{Z}[u_1^{\pm1}]$-modules); under this identification, the relations given by the vanishing of $f(u_1,u_2) = f_0u_2^{n_0} + f_1u_1u_2^{n_1} + \cdots + f_du_1^d$ become

$$-u_1^dP_d = P_0 + u_1P_1 + \cdots + u_1^{d-1}P_{d-1} \tag{2.13}$$

where P_i is the N-circulant associated to $f_i u_2^{n_i}$ (by our writing of f, $n_d = 0$, so the matrix P_d is diagonal). It follows that the automorphism $\beta_{N,(1,0)}$ of the solenoid $X_{\mathfrak{M}}^{N,(1,0)}$ is given explicitly as one in the form of Lawton,

$$X_{\mathfrak{M}}^{N,(1,0)} = \{\mathbf{x} \in (\mathbb{T}^N)^{\mathbb{Z}} |\, (\mathbf{x}_k, \dots, \mathbf{x}_{k+(d-1)})P = (\mathbf{x}_{(k+1)}, \dots, \mathbf{x}_{k+d})Q\}$$

with the automorphism given by the left shift, where

$$P = \begin{bmatrix} 0 & I & 0 & & \\ 0 & 0 & I & 0 & \\ & & & \ddots & \\ & & & & I \\ P_0 & P_1 & \dots & & P_{d-1} \end{bmatrix} \tag{2.14}$$

and

$$Q = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & -P_d \end{bmatrix}. \tag{2.15}$$

Thus, $\mathbb{P}(\beta_{N,(a,b)})$ is contained in the set of places for which some eigenvalue of the rational matrix

$$A = Q^{-1}P = \begin{bmatrix} 0 & I & 0 & & \\ 0 & 0 & I & 0 & \\ & & & \ddots & \\ & & & & I \\ R_0 & R_1 & \dots & & R_{d-1} \end{bmatrix} \tag{2.16}$$

is not a unit (here each R_i is an $N \times N$ circulant with at most one rational entry per row). For each $N \times N$ circulant matrix R_i there is an associated monomial $r_i \in \mathbb{Q}[x]/\langle 1 - x^N \rangle$. This assignment is functorial: the matrix R_i is multiplication by $r_i(x)$ on $\mathbb{Q}[x]/\langle 1 - x^N \rangle$ with the canonical basis, so the polynomial corresponding to the product of two circulants is the product of the polynomials. The eigenvalues of R_i are the zeros of $\prod_{j=1,\dots,N}(r_i(e^{2\pi ij/N}) - \lambda)$. Thus, we may identify the matrix A with the linear map

$$\begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ & & & \ddots & \\ & & & & 1 \\ r_0 & r_1 & \dots & & r_{d-1} \end{bmatrix} \tag{2.17}$$

on $\left(\mathbb{Q}[x]/\langle 1-x^N\rangle\right)^d$. To find the eigenvalues of (2.16), first find the formal eigenvalues of (2.17), that is, solve the equation

$$r_0(x)+r_1(x)\lambda(x)+r_2(x)\lambda(x)^2+\cdots+r_{d-1}(x)\lambda(x)^{d-1}=\lambda(x)^d \tag{2.18}$$

in algebraic functions $\lambda(x)$. For each of the d solutions $\lambda_1(x),\dots,\lambda_d(x)$ to (2.18), solve the equation

$$\prod_{j=1,\dots,N}(\lambda_k(e^{2\pi ij/N})-\lambda)=0 \tag{2.19}$$

for λ. Thus the eigenvalues of (2.16) are the solutions of

$$\prod_{k=1,\dots,d}\prod_{j=1,\dots,N}(\lambda_k(e^{2\pi ij/N})-\lambda)=0. \tag{2.20}$$

(In (2.19) and (2.20), i denotes the usual square root of -1.) Expanding (2.20) gives a polynomial with rational coefficients of degree $d\times N$, whose coefficients are symmetric functions in the $\lambda_k(e^{2\pi ij/N})$. Evaluating (2.18) on the unit root $e^{2\pi ij/N}$ gives a polynomial equation with algebraic coefficients satisfied by $\lambda_k(e^{2\pi ij/N})$. Each coefficient $r_\ell(e^{2\pi ij/N})$ in this polynomial has $|r_\ell(e^{2\pi ij/N})|_p$ less than or equal to 1 for all but finitely many primes p. The exceptional primes are those dividing the denominators of the r_ℓ's or the determinant of P_d, and this is clearly a finite set. It follows that $\log^+|\lambda_k(e^{2\pi ij/N})|$ is zero for all but finitely many primes p, independently of N. ∎

3. Stability: cyclic case

It is not immediately clear from the definitions how to compute the oriented local entropies (equivalently, how to find the zeros of the principal associated primes of $\mathfrak{R}/\langle f, 1-u_1^{Nb}u_2^{-Na}\rangle$ as a $\mathbb{Z}[(u_1^a u_2^b)^{\pm1}]$-module) without passing to the (a^2+b^2) power of the system. We describe one method here. Given a $\mathbb{Z}^2$ action α, and a matrix $B\in \mathrm{SL}(2,\mathbb{Z})$, define a new action ${}^B\alpha$ by setting

$${}^B\alpha_{\mathbf{n}}=\alpha_{\mathbf{n}B^{-1}}. \tag{3.1}$$

If $\alpha=\alpha^{\mathfrak{R}/\langle f\rangle}$ then ${}^B\alpha=\alpha^{\mathfrak{R}/\langle {}^Bf\rangle}$ where ${}^Bf(u_1,u_2)=f(\mathbf{u}^{(1,0)B^{-1}},\mathbf{u}^{(0,1)B^{-1}})$. If (a,b) is a direction in $\mathbb{Z}^2$ then we may find a matrix $B\in\mathrm{SL}(2,\mathbb{Z})$ for which

$${}^B\alpha_{(a,b)}=\alpha_{(1,0)} \tag{3.2}$$

since a and b are coprime.

Thus, in finding $\beta_{N,(a,b)}$ we may assume without loss of generality that $(a,b) = (1,0)$. However, the change of variables will not in general send $(b,-a)$ to $(0,1)$ of course; let us assume that the change of variables sends $(b,-a)$ to (c,d).

Fix $M = \mathfrak{R}/\langle f\rangle$, where f is a non-constant polynomial, and consider the map $\alpha^M_{(1,0)}$ restricted to points of period $N(c,d)$, $d \neq 0$. As $\mathbb{Z}[u_1^{\pm 1}]$-modules,

$$\frac{\mathfrak{R}}{\langle 1-(u_1^c u_2^d)^N\rangle} \cong \bigoplus_{j=1,\dots,N(c^2+d^2)} u_2^j \mathbb{Z}[u_1^{\pm 1}] \cong \mathbb{Z}[u_1^{\pm 1}]^{N(c^2+d^2)}. \tag{3.3}$$

It follows that

$$\frac{\mathfrak{R}}{\langle f, 1-(u_1^c u_2^d)^N\rangle} \cong \frac{\mathbb{Z}[u_1^{\pm 1}]^{N(c^2+d^2)}}{A\mathbb{Z}[u_1^{\pm 1}]^{N(c^2+d^2)}} \tag{3.4}$$

(again, as $\mathbb{Z}[u_1^{\pm 1}]$-modules) where A is the matrix whose j^{th} row comprises the coefficients of the polynomial $u_1^j f(u_1,u_2)$ (reduced using the relation $(u_1^c u_2^d)^N = 1$) and written out in ascending powers of u_2 with coefficients in $\mathbb{Z}[u_1^{\pm 1}]$.

THEOREM 3.1: *Let $\mathfrak{M} = \mathfrak{R}/\langle f\rangle$, where f is a non-constant irreducible polynomial. Then, for any direction (a,b) inside an associated cone of f, $X^{N,(a,b)}$ is connected. Moreover, the (a,b)-oriented local entropies exist, and the decomposition of $h(\alpha^{\mathfrak{M}})$ into (a,b)-oriented local entropies,*

$$h(\alpha^{\mathfrak{M}}) = \sum_{p\le\infty} h_p^{(a,b)}(\alpha^{\mathfrak{M}}),$$

is constant as (a,b) varies in each associated cone of f.

That is, if (a,b) and (a',b') are directions in the same cone of f, then, for each p, $h_p^{(a,b)}(\alpha^{\mathfrak{R}/\langle f\rangle}) = h_p^{(a',b')}(\alpha^{\mathfrak{R}/\langle f\rangle})$.

Proof: First consider connectedness: this follows from the proof of Lemma 2.6 in which (a Cartesian power of) $X_{N,(a,b)}$ is explicitly identified as a solenoid; we give a different direct proof here. The dual of the group $X^{N,(a,b)}$ is the $\mathfrak{R}$-module

$$\mathfrak{W} = \frac{\mathfrak{R}}{\langle f, 1-(u_1^{-b}u_2^a)^N\rangle}. \tag{3.5}$$

In order to show that $X^{N,(a,b)}$ is connected, it is therefore enough to show that $\mathfrak{W}$ is torsion-free as an additive group. This in turn is guaranteed if there are no polynomials $p, q \in \mathfrak{R}$ for which $h(u_1, u_2) = pf + q(1 - (u_1^{-b}u_2^{a})^N)$ is a constant. Now if h is constant, then for every $z \in \mathbb{C}$,

$$f(z^{Nb}, z^{Na}) = Cz^k \tag{3.6}$$

for some k, provided that $\{(0,0), (-Nb, Na)\}$ is not in the support of f. The left hand side of (3.6) is a polynomial in z^N since a and b have no common factor, so k is a multiple of N. Writing out $f(u_1, u_2)$ as $\sum u_2^j f_j(u_1)$ we get

$$f_0(z^{Nb}) + z^{Na}f(z^{Nb}) + \cdots + f_d(z^{Nb})z^{dNa} = Cz^{cN}. \tag{3.7}$$

Since a and b are coprime, we can read off all the coefficients of f from (3.7), which is absurd. We deduce that $\mathfrak{W}$ is torsion-free so $X^{N,(a,b)}$ is connected.

Now consider the convergence of (2.7). By (2.8), it is enough to establish this for all the finite places. By the remarks following (2.10), it is enough to show this for the map $\beta^{\mathfrak{L}}_{N,(1,0)}$ (in the notation of Lemma 2.5; notice that the normalization factor $\frac{1}{a^2+b^2}$ in (2.7) comes from (2.10)). That is, we may assume that $(a, b) = (1, 0)$. Now in (2.18) the monomials $r_\ell(x)$ are independent of N, so the d algebraic functions $\lambda_1(x), \ldots, \lambda_d(x)$ are also independent of N. Let $\lambda_{k,j} = \lambda_k(e^{2\pi i j/N})$, for $k = 1, \ldots, d$ and $j = 1, \ldots, N$. Then we claim that

$$\frac{1}{N}\sum_{k=1}^{d}\sum_{j=1}^{N}\log^+|\lambda_{k,j}|_p \tag{3.8}$$

converges as $N \to \infty$. Now by Lemma 2.2, the map $\beta^{\mathfrak{L}}_{N,(1,0)}$ is expansive. It follows that each λ_k does not vanish on the unit circle (if λ_k vanished at $e^{2\pi i s}$ then $e^{2\pi i s}$ would have to lie in the closure of $\{z \in \mathbb{C} | \; f(z, w) = 0$ for some unit root $w\}$; by [S] expansiveness requires that this set miss the unit circle). Let U denote the p-adic unit circle. This is the closure of the group of algebraic roots of unity inside Ω_p, the smallest field containing $\mathbb{Q}_p$ and all the algebraic numbers. Each of the λ_k extends to a continuous function on U. Now U is a locally compact group and $\log^+|\lambda_k|_p$ is continuous on U, so we may take the expression in (3.8) as a Riemann sum for the p-adic integral (see [H], Chapter 4 for the details). This shows that the contribution at p can be written

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{d}\sum_{j=1}^{N}\log^+|\lambda_{k,j}|_p = \sum_{k=1}^{d}\int_U \log^+|\lambda_k(e^{2\pi i s})|_p d\mu. \tag{3.9}$$

Notice that in (3.9), μ denotes Haar measure on the locally compact group U.

Finally, consider the effect of changing (a,b) to (a',b') in the same associated cone of f. Let the associated matrices be

(3.10)
$$A = \begin{bmatrix} 0 & I & 0 & & \\ 0 & 0 & I & 0 & \\ & & & \ddots & \\ & & & & I \\ R_0 & R_1 & \dots & & R_{d-1} \end{bmatrix}; \quad A' = \begin{bmatrix} 0 & I & 0 & & \\ 0 & 0 & I & 0 & \\ & & & \ddots & \\ & & & & I \\ R'_0 & R'_1 & \dots & & R'_{d-1} \end{bmatrix}$$

as in (2.16). Since we are in the same cone, the matrices R_i and R'_i differ only in that R_i is the N-circulant of $r_i(x) = r_i x^j$, while R'_i is the N-circulant of the polynomial $r'_i(x) = r_i x^{j'}$, where π: $j \mapsto j'$ is some fixed permutation. Now let $\lambda_{(k,j)} = \lambda'_k(e^{2\pi ij/N})$ be the eigenvalues of A'; these numbers may be obtained as follows. For each $j = 1, \dots, N$, solve (2.18) for λ with $x = e^{2\pi ij/N}$. Now the Newton polygon of (2.18) with x set equal to any unit root does not change if each r_i is replaced by r'_i, so the p-adic size of the eigenvalues is not changed. By (2.8) the same holds for the infinite place. ∎

4. Oriented entropies: general expansive case

Now consider an expansive $\mathbb{Z}^2$ action α by automorphisms of a connected compact group X. By [S], we may assume that $\alpha = \alpha^{\mathfrak{M}}$ for a Noetherian $\mathfrak{R}$-module $\mathfrak{M}$ with the property that $\alpha^{\mathfrak{R}/\mathfrak{p}}$ is expansive and $X_{\mathfrak{R}/\mathfrak{p}}$ is connected, for each prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$. Choose a prime filtration

$$\mathfrak{M} = \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \cdots \supset \mathfrak{M}_{s-1} \supset \mathfrak{M}_s = \{0\} \tag{4.1}$$

of $\mathfrak{M}$ [S]. Successive quotients $\mathfrak{M}_j/\mathfrak{M}_{j+1}$ are isomorphic as $\mathfrak{R}$-modules to cyclic modules of the form $\mathfrak{R}/\mathfrak{q}_j$, with $\mathfrak{q}_j$ a prime ideal containing an associated prime of $\mathfrak{M}$. By [LSW], the entropy of $\alpha^{\mathfrak{M}}$ is given by a finite sum

$$h(\alpha^{\mathfrak{M}}) = \sum_{j=0}^{s-1} h(\alpha^{\mathfrak{R}/\mathfrak{q}_j}). \tag{4.2}$$

In (4.2), each term is then either zero (if $\mathfrak{q}_j$ is non-principal) or a Mahler measure of the form (1.6).

Definition 4.1: Let $\alpha = \alpha^{\mathfrak{M}}$ be an expansive $\mathbb{Z}^2$ action by automorphisms of a compact connected abelian group. The **associated cones** of α are the open cones in $\mathbb{Z}^2$ with vertex at the origin whose edges are lines orthogonal to lines joining any pair of points in the support of any one of the polynomials $f_1, \dots, f_t$, where $\langle f_1\rangle, \dots, \langle f_t\rangle$ are the principal ideals appearing in a prime filtration of $\mathfrak{M}$.

That is, given the module $\mathfrak{M}$, form a filtration of the form (4.1). Each ideal $\mathfrak{q}_j$ that is principal gives rise to finitely many lines in the sense of Definition 2.4: the cones associated to $\alpha^{\mathfrak{M}}$ are then the open cones defined by the union of all those lines.

Definition 4.2: For any direction (a, b), the (a, b)**-oriented local entropies** of $\alpha^{\mathfrak{M}}$ are the quantities (in terms of (4.1))

$$h_p^{(a,b)} = \sum_{j:\mathfrak{q}_j \text{ principal}} h_p^{(a,b)}(\alpha^{\mathfrak{R}/\mathfrak{q}_j}),$$

where the summands are defined by Definition 2.3. ∎

THEOREM 4.3: *Let $\alpha = \alpha^{\mathfrak{M}}$ be an expansive $\mathbb{Z}^2$ action by automorphisms of a compact connected abelian group. Then, for any direction (a, b) inside an associated cone of α, there is an oriented local decomposition of the entropy*

$$h(\alpha^{\mathfrak{M}}) = \sum_{p\le\infty} h_p^{(a,b)}(\alpha^{\mathfrak{M}}).$$

The decomposition is constant as (a, b) varies inside an associated cone of α.

Proof: Notice that since $\mathfrak{M}$ is torsion-free as an additive group, the same is true of each succesive quotient $\mathfrak{M}_j/\mathfrak{M}_{j+1}$ for which $\mathfrak{q}_j$ is principal: if $\mathfrak{q}_j$ is a principal prime ideal containing an associated prime of $\mathfrak{M}$, then $\mathfrak{q}_j$ is itself an associated prime of $\mathfrak{M}$. On the other hand, if $\mathfrak{q}_j$ is an associated prime of $\mathfrak{M}$, then there is an injective $\mathfrak{R}$-module homomorphism $\mathfrak{R}/\mathfrak{q}_j \to \mathfrak{M}$, dual to which is a surjective homomorphism from the connected group $X_{\mathfrak{M}}$ onto $X_{\mathfrak{R}/\mathfrak{q}_j}$; it follows that $X_{\mathfrak{R}/\mathfrak{q}_j}$ is connected.

By Definition 4.1 and Theorem 3.1, the summands in Definition 4.2 are all defined. By Theorem 2.5 we therefore have, for each j with $\mathfrak{q}_j$ principal,

$$h(\alpha^{\mathfrak{R}/\mathfrak{q}_j}) = \sum_{p\le\infty} h_p^{(a,b)}(\alpha^{\mathfrak{R}/\mathfrak{q}_j}). \tag{4.3}$$

On the other hand, by [LSW],

$$h(\alpha^{\mathfrak{M}}) = \sum_j h(\alpha^{\mathfrak{R}/\mathfrak{q}_j}) = \sum_{j:\mathfrak{q}_j \text{ principal}} h(\alpha^{\mathfrak{R}/\mathfrak{q}_j}) \tag{4.4}$$

(see Definition 4.2). It follows that, for a direction (a, b) within an associated cone of α,

$$h(\alpha^{\mathfrak{M}}) = \sum_{j:\mathfrak{q}_k \text{ principal}} \sum_{p\le\infty} h_p^{(a,b)}(\alpha^{\mathfrak{R}/\mathfrak{q}_j}) = \sum_{p\le\infty} h_p^{(a,b)}(\alpha^{\mathfrak{M}}) \tag{4.5}$$

by Definition 4.2.

The stability of the decomposition within associated cones is a direct consequence of Theorem 3.1. ∎

Remark: Since our primary goal is simply to exhibit a decomposition into oriented local entropies, Definition 4.2 is the simplest possible approach. An alternative is the following. Let $\mathfrak{M}$ be the $\mathfrak{R}$-module corresponding to the expansive $\mathbb{Z}^2$ action α. Then Definition 2.1 applies directly, producing for each direction (a, b) and integer $N \ge 1$ an automorphism of a solenoid $\beta_{N,(a,b)}$. The oriented local entropies may then be defined exactly as in Definition 2.3. However, to be sure that the limit in Definition 2.3 exists, one needs to reduce to the cyclic case. To see that this approach gives the same decomposition as exhibited in Theorem 4.3, consider the chain of $\mathbb{Z}[u_1^a u_2^b]$-modules

$$\frac{\mathfrak{M}}{\langle 1-u_1^{Nb}u_2^{-Na}\rangle\mathfrak{M}} \supset \frac{\mathfrak{M}_1}{\langle 1-u_1^{Nb}u_2^{-Na}\rangle\mathfrak{M}_1} \supset\cdots\supset \frac{\mathfrak{M}_{s-1}}{\langle 1-u_1^{Nb}u_2^{-Na}\rangle\mathfrak{M}_{s-1}}. \tag{4.6}$$

The chain (4.6) is not a prime filtration of the $\mathbb{Z}[u_1^a u_2^b]$-module

$$\mathfrak{M}/\langle 1-u_1^{Nb}u_2^{-Na}\rangle\mathfrak{M}$$

(for example, as we have seen in Section 3, if $\mathfrak{M}$ is a cyclic $\mathfrak{R}$-module, then $\mathfrak{M}/\langle 1-u_1^{Nb}u_2^{-Na}\rangle\mathfrak{M}$ is in general a non-cyclic $\mathbb{Z}[u_1^a u_2^b]$-module.) Nonetheless, the jth succesive quotient in (4.6) corresponds to the dynamical system obtained by restriciting to points of period $(Nb, -Na)$ in the system corresponding to the $\mathfrak{R}$-module $\mathfrak{R}/\mathfrak{q}_j$. That is, if the $\mathfrak{R}$-module filtration is taken before restricting to periodic points (as in Definition 4.2) the oriented local decomposition of the entropy obtained is exactly the same as that obtained by first restricting to periodic points and then making a $\mathbb{Z}[u_1^a u_2^b]$-module filtration, as in this Remark.

5. Examples and remarks

To clarify Sections 3 and 4 we give some examples.

Example 5.1: Let $f(u_1, u_2) = 3u_1 - 2$, and consider the direction $(a,b) = (2,1)$. We need to understand the map $\times u_1^2 u_2$ on the module

$$\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 3u_1 - 2, 1 - (u_1 u_2^{-2})^N \rangle}.$$

Apply the change of variables matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This makes ${}^B f(u_1, u_2) = 3u_1 u_2^{-1} - 2$, and transforms $u_1^2 u_2$ into u_1 and $u_1 u_2^{-2}$ into $u_1^3 u_2^{-5}$. We can therefore identify $\beta_{N,(2,1)}$ with the map dual to multiplication by u_1 on the $\mathbb{Z}[u_1^{\pm 1}]$-module

$$\begin{aligned}\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 3u_1 u_2^{-1} - 2, 1 - (u_1^3 u_2^{-5})^N \rangle} &\cong \frac{\mathbb{Z}[u_1^{\pm 1}] \oplus u_2 \mathbb{Z}[u_1^{\pm 1}] \oplus \cdots \oplus u_2^{5N-1} \mathbb{Z}[u_1^{\pm 1}]}{L} \\ &\cong \frac{\mathbb{Z}[u_1^{\pm 1}]^{5N}}{A \mathbb{Z}[u_1^{\pm 1}]^{5N}}\end{aligned}$$

where (notice that $u_1 u_2^{-1} = u_1^{-3N+1} u_2^{5N-1}$)

$$A = \begin{bmatrix} -2 & 0 & & & 3u_1^{-3N+1} \\ 3u_1^{-3N+1} & -2 & 0 & & \\ & & \ddots & & \\ & & 3u_1^{-3N+1} & -2 & \\ & & & 3u_1^{-3N+1} & -2 \end{bmatrix}$$

Then

$$\det(A) = \prod_{j=1,\ldots,5N} (-2 + (3u_1^{-3N+1}) e^{2\pi i j/5N}).$$

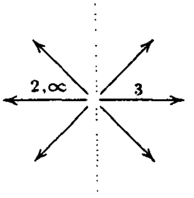
It follows that the eigenvalues of the rational matrix corresponding to $\beta_{N,(a,b)}$ are given by

$$\lambda_\ell = \left(\sqrt[3N-1]{\frac{2}{3}} \right) \epsilon_\ell,$$

where ϵ_ℓ are unit roots, and $\ell = 1, \ldots, 5N(3N-1)$. Thus, the oriented entropy in this direction is all 3-adic, and

$$h_\infty^{(2,1)}(\alpha^M) = 0; \quad h_2^{(2,1)}(\alpha^M) = 0; \quad h_3^{(2,1)}(\alpha^M) = \log 3.$$

Similar calculations yield the following picture for the local entropies in the system:



In this picture the dotted line represents the line separating the two open cones associated to the polynomial.

Example 5.2: In order to illustrate what may happen in a direction separating the associated cones, consider the direction $(a,b) = (0,1)$ for the system considered in Example 4.1. The map is now $\times u_2$ on the module

$$\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 3u_1 - 2, 1 - u_1^N \rangle}.$$

Notice that

$$\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 1 - u_1^N \rangle} \cong \bigoplus_{j=0,\ldots,N-1} u_i^j \mathbb{Z}[u_2^{\pm 1}] \cong \mathbb{Z}[u_2^{\pm 1}]^N,$$

and under this isomorphism of $\mathbb{Z}[u_2^{\pm 1}]$-modules, the ideal $\langle 3u_1 - 2 \rangle$ is sent to $A\mathbb{Z}[u_2^{\pm 1}]^N$, where A is the circulant matrix

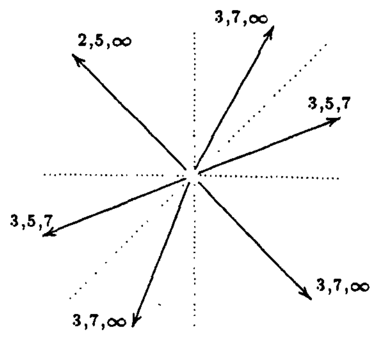
$$A = \begin{bmatrix} -2 & 3 & & & \\ 0 & -2 & 3 & & \\ & & \ddots & & \\ & & 0 & -2 & 3 \\ 3 & & & & -2 \end{bmatrix}.$$

Thus, in an obvious sense

$$\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 3u_1 - 2, 1 - u_1^N \rangle} \cong \left(\frac{\mathbb{Z}^N}{A\mathbb{Z}^N} \right) [u_2^{\pm 1}]$$

as $\mathbb{Z}[u_2^{\pm 1}$-modules, so the compact group automorphism dual to multiplication by u_2 on $\frac{\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]}{\langle 3u_1 - 2, 1 - u_1^N \rangle}$ is (topologically conjugate to) the full shift on $|\det(A)| = |(-3)^N - 2^N|$ symbols. Algebraically, the map is the full shift with alphabet given by the finite $\mathbb{Z}^N / A\mathbb{Z}^N$.

Example 5.3: Let $f(u_1,u_2) = 10+21u_1+105u_2$, so $h(\alpha^{\mathfrak{R}/\langle f\rangle}) = \log 105$. Similar calculations yield the following portrait of the local entropy contributions.



In the diagram the three dotted lines represent the lines separating the six open cones associated with the polynomial.

We close with some problems and remarks.

[1] We have restricted attention to expansive actions on connected groups. By a result of Kitchens and Schmidt (see [KS]), such actions can only be carried on abelian groups so the abelian assumption is not really needed in Theorem 3.1 and 4.3.

[1.1] The assumption of expansiveness cannot be relaxed totally: consider the $\mathbb{Z}^2$ action $\alpha^{\mathfrak{M}}$ where

$$\mathfrak{M} = \bigoplus_{(c,d)\in\mathbb{Z}^2} \mathfrak{R}/\langle 3u_1^c u_2^d - 2\rangle.$$

Then it is clear (cf. Example 5.1) that (a,b)-oriented entropy is being exchanged between the 3-adic and the 2-adic and infinite components whenever the direction (a,b) crosses a rational line. The oriented entropies in this example are all either infinity or zero.

A more subtle form of non-expansiveness arises when the associated module is still Noetherian, but the action is non-expansive (in the language of [KS], these are actions with the Descending Chain Condition but without expansiveness; in the case of a $\mathbb{Z}$ action on the torus these are the quasihyperbolic actions considered in [Li]). For these systems, there are enough periodic points (they are dense by [KS]), and it is conjectured that a version of (2.2) holds (suitably altered to accommodate periods for which there are infinitely many periodic points). In an important special case, namely the dynamical system corresponding to the module $\mathfrak{R}/\langle 1+u_1+u_2\rangle$, the local entropies can be computed by avoiding periods contained in $3\mathbb{Z}\times 3\mathbb{Z}$. For this example, it is found that every direction for which the local entropies are defined has only Archimedean hyperbolicity, so the decomposition (1.10) is trivial. The module $\mathfrak{R}/\langle 4-u_1-u_2-u_1^{-1}-u_2^{-1}\rangle$ gives rise to a system with infinitely many points of each period. This means that the map $\beta_{N,(a,b)}$ (as in Section 2.6) is always non-ergodic. While it is clear what oriented local entropy decomposition to expect in such a situation, our methods break down because the convergence of the growth rate of periodic points (see Section 7 of [LSW]) is no longer available. This is a weakness of our method rather than a reflection of the underlying entropy contributions from p-adic and geometric hyperbolicities.

[1.2] The assumption of connectedness is of a different nature, and may be colloquially put as follows: there are no interesting entropies on zero-dimensional groups. If $\alpha=\alpha^{\mathfrak{M}}$ is an expansive $\mathbb{Z}^2$ action on a disconnected group $X=X_{\mathfrak{M}}$, then $\mathfrak{M}$ has an associated prime $\mathfrak{p}$ with the property that $p\in\mathfrak{p}$ for some rational prime p. If $\mathfrak{p}$ is non-principal, then by [LSW], the corresponding summand in (4.2) is zero. If $\mathfrak{p}$ is principal, then $\mathfrak{p}=\langle p\rangle$, and the corresponding dynamical system is simply the full shift on p symbols. There does not seem to be any meaningful sense in which the entropy of such a system may be decomposed into oriented local contributions: in particular, such a dynamical system looks exactly the same when viewed from any orientation.

[2] If $\alpha^{\mathfrak{M}}$ has positive entropy, then for every $(a,b)\neq(0,0)$, the automorphism $\alpha^{\mathfrak{M}}_{(a,b)}$ has infinite entropy by a result of Conze ([C]), and is naturally given by an infinite matrix of Toeplitz type. Can the compact spectrum of the associated operator be used to give the local entropies? If so, this may remove the difficulties encountered in trying to extend Theorem 4.3 to the non-expansive case.

[3] Passing from $\mathbb{Z}^2$ actions to $\mathbb{Z}^d$ actions, $d>2$, is reasonably straightforward by an inductive argument, but involves some complications. The geometric structures corresponding to the associated cones of polynomials in two

variables need to be described, and the change of variable method becomes much more involved (to produce the matrix used to rotate the chosen orientation to an axis amounts to finding a convenient element of $\mathrm{SL}(d,\mathbb{Z})$ given a first column of integers with no common factor).

[4] The actions we have considered have positive global entropy (that is, positive entropy as $\mathbb{Z}^2$ actions) and they are therefore expansive as actions, but the individual elements are not expansive. At the opposite extreme, one may consider $\mathbb{Z}^d$ actions generated by d commuting automorphisms of a compact connected group X, each of which is expansive. It follows that X is a solenoid ([L]), and for $d > 1$ the action generated has zero global entropy. Actions of this kind are examples of principal Anosov actions in the terminology of [KSp]. In [KSp], a notion of non-Archimedean Lyapunov exponents is developed for such actions. We close with the following heuristic observation: Katok and Spatzier exhibit interesting rigidity phenomena for commuting Anosov maps (of which commuting expansive solenoidal automorphisms are examples) quite different to the case of single maps. The actions we have considered here, which as actions only are expansive, have positive entropy, and so on, are expected to not exhibit any of these rigidity phenomena, and in this sense to be similar to actions by single automorphisms.

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